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4801. Assume, for a contradiction, that $\log_a b$ is rational, and can be expressed, for $p, q \in \mathbb{Z}$, as $\log_a b = {}^p/_q$. Exponentiate both sides over base a, and raise both sides to the power q. Consider prime factors.

4802. Call the upwards acceleration of the pulley a. If the bob of mass m_1 remains at rest, then the bob of mass m_2 must be accelerating upwards at $2a$. Beyond this, it's a standard pulley problem. The accelerations of the bobs are now different from each other though, so you can't shortcut with a single equation of motion for the whole system.

4803. Let the functions have definitions

$$\begin{aligned} \mathrm{f}(x) &= ax + b \\ \mathrm{g}(x) &= cx + d. \end{aligned}$$

Simplify $\mathrm{f}^2(x)$ and then $\mathrm{f}^3(x)$. State a matching result for $\mathrm{g}^3(x)$ and then set up an identity. Equate coefficients to show that $a = c$. Sub this into an equation of the constant terms. Factorise and show that the quadratic factor has $\Delta < 0$.

4804. Write the equation of the graph out longhand: it has single asymptotes at $x = -3, -2, ..., 2, 3$. Work out the sign of the y either side of each asymptote, by considering the sign of the dominating term.

4805. Let the outer pentagon have side length 1. Work through the geometry of the isosceles triangles to show that the side of the inner pentagon is $\sec 36° \sin 18°$. You now need to show that $\sec 36° = 4 \sin 18°$. Rearrange the following to make $\sec 36°$ the subject:

$$\sin 72° = 2 \sin 36° \cos 36°.$$

Manipulate with identities.

4806. Use the substitution $u = \ln x$. Then integrate by parts, using the tabular integration method.

4807. Prove this by construction, i.e. explicitly find a set of transformations that will do the job. Let the x intercepts be $\{b - d, b, b + d\}$. Work out the set of transformations (in the x direction) which will take this set onto the set $\{-1, 0, 1\}$. Consider the y direction also.

4808. (a) Use $\mathrm{c}_2(x) = 1 - \frac{1}{2}x^2$.

(b) Set up the model $\cos(x) - \mathrm{c}_2(x) = kx^4$. Take lns of both sides and simplify. You need to show (by drawing a line of best fit), that there is an approximately linear relationship between $\ln\big(\cos(x) - \mathrm{c}_2(x)\big)$ and $\ln x$.

(c) Comparing the measured equation of the line of best fit to the linear relationship derived in part (b).

4809. The relevant substitution, which is an instance of the chain rule, is

$$a = v\frac{dv}{dx}.$$

Having used this, you can separate the variables.

4810. Reflection in the line $y = x + c$ is equivalent to reflection in the line $y = x$, followed by translation by vector $-c\mathbf{i} + c\mathbf{j}$.

4811. (a) Differentiate and use $y - y_1 = m(x - x_1)$.

(b) Consider the fact that two adjacent branches of $y = \tan x$ have rotational symmetry around the point $({}^\pi/_2, 0)$.

(c) Substitute $({}^\pi/_2, 0)$ into the equation in part (a). The equation is not analytically solvable. Set up the Newton-Raphson iteration (or fixed-point iteration), and solve numerically for p. Then calculate the distance using Pythagoras.

4812. Draw the angle bisectors at A and B, naming their point of intersection X. Drop perpendiculars from X to all three sides. Use two pairs of congruent triangles to prove the congruency of a third pair.

4813. Complete the square in the denominator. Then let $x + 4 = \tan\theta$.

4814. If $y = \mathrm{g}'(x)$ has rotational symmetry around (a, b), then $\mathrm{g}'(a + x)$ and $\mathrm{g}'(a - x)$ are equidistant from b. Put this into algebra, and integrate it.

4815. The relevant distance is between the centre of the tetrahedron and the centre of one of the its faces. Place the face in question flat to the ground. The task is then to find the height of the centre of the tetrahedron. Use trigonometry.

— Nota Bene —

You may want, as a shortcut, to use an analogous 3D result to the following: that the centroid of a triangle lies $\frac{2}{3}$ of the way along its medians.

4816. (a) Expand each factor with a compound-angle formula. You should get a difference of two squares. Simplify this, and then use the first Pythagorean identity, followed by a double-angle formula.

(b) Replace $\cos 6x - \cos 2x$ using the proved sum-to-product formula. You should get a common factor of $\sin 4x$. Factorise and solve.

4817. You can rule out a pair of possibilities based on symmetry. Then test the origin to choose between the remaining two.

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4818. Consider C_3 and C_4, classifying the result by the number of heads attained in C_3:

C_3	C_4	Probability
0	$1, 2, 3, 4$	$\frac{1}{8} \times \frac{15}{16} = \frac{15}{128}$
1	$2, 3, 4$	$\frac{3}{8} \times \frac{11}{16} = \frac{33}{128}$
2	$3, 4$	$\frac{3}{8} \times \frac{5}{16} = \frac{15}{128}$
3	4	$\frac{1}{8} \times \frac{1}{16} = \frac{1}{128}$

The key point is the symmetry above and below the dashed line. Add up the probabilities in the right-hand column, and prove that the result you get generalises.

4819. Find the equations of the normals, with coefficients in terms of p. Eliminate y, substitute $x = \frac{15}{2}$, and solve for p.

4820. (a) Consider the fact that adding $\mathrm{g}(x)$ to $\mathrm{f}(x)$ will also add $\mathrm{g}(x)$ to the equation of any tangent line to $y = \mathrm{g}(x)$.

(b) Let $\mathrm{g}(x) = -x - 1$.

4821. Set up the cosine rule:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

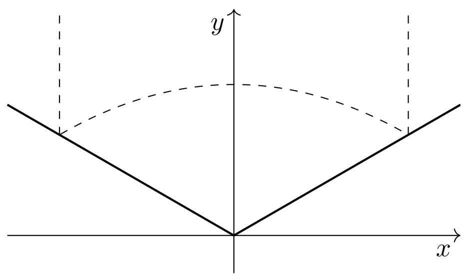
Simplify an expression for $\sin C = \sqrt{1 - \cos^2 C}$. Substitute this into $A_\triangle = \frac{1}{2}ab \sin C$ and massage the resulting algebra.

4822. The key fact is this: if two cubics have the same set of roots, then they must be scalar multiples of one another. Hence, the derivative of the quartic must be

$$\frac{dy}{dx} = k(ax^3 + bx^2 + cx + d).$$

4823. Integrate by parts with $u = x$ and $v' = e^x \cos x$. You'll need to perform another integration by parts in order to find v.

4824. The motion looks like this:



Find the point at which the projectile bounces, and the direction in which it is travelling when it does so. Use these and the angle of inclination of the surfaces to show that the above is the case.

4825. By the chain rule,

$$\frac{dy_n}{dy_1} \equiv \frac{dy_n}{dy_{n-1}} \times \frac{dy_{n-1}}{dy_{n-2}} \times ... \times \frac{dy_2}{dy_1}$$

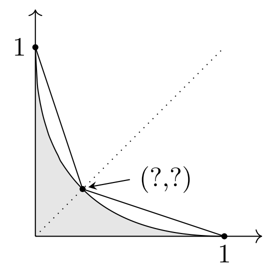
Factorise the given formula before substituting it into the above. That way, most of the factors will cancel. Then integrate with respect to y_1.

4826. The middle right must be $\times$. From there, classify the grids by the number of $\bigcirc$ in the top row.

4827. Consider $y = x^{-1.8} \equiv x^{-\frac{9}{5}}$. Since, both 9 and 5 are odd, this is a curve akin to $y = x^{-1}$ and $y = x^{-3}$, with gradients in between the two. Sketch this, then $y = 1 - x^{-1.8}$, then $y = \sqrt{1 - x^{-1.8}}$.

4828. The long method involves finding the probability that $\{3, 2, 1\}$ couples sit together. The individual probabilities have appeared earlier in this book as separate problems. But there is an easier way to do it, which might seem a little like magic. Find the expectation of "Couple A sitting together", i.e. *Across a large number of trials, in what fraction of trials are couple A sitting together?* This applies symmetrically for the other couples. The numbers combine very simply.

4829. The following diagram shows A_1. Bound the area A_1 (and A_n in general) by the area below and to the left of the line segments:



Start by finding the coordinates of the point.

4830. Express the equation of P_2 algebraically, enacting the enlargement by replacing x by $2x$ and y by $2y$. Then use the discriminant to ensure that the two parabolae are tangent.

4831. (a) Differentiate by first writing $\log_k e$ as $\frac{\ln x}{\ln k}$.

(b) Differentiate again and show that the second derivative is negative, noting that, since $k > 1$, $\ln k > 0$.

(c) The result from (b) implies that the curve $y = \log_k x$ is bounded above by its tangent line at $x = 1$. So, the area of the region under the curve is bounded above by the area of the region under its tangent line. This region is a right-angled triangle.

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4832. (a) Differentiate implicitly, take out a factor of $\frac{dy}{dx}$ and rearrange.

 (b) Set the numerator to zero, and substitute for x in the original equation.

 (c) Solve the equation in (b) using a double-angle formula, showing that it produces infinitely many y values. Show that each produces an x value, giving infinitely many SPs.

4833. We are given that the sum of the four scores is 12. We are looking for the product to be 64, which is 2^6. So, each score X_i must be $1, 2$ or 4. There is only one combination of four of these which adds to 12:

$$4+4+2+2=12.$$

You need to find

$$\mathbb{P}\left(\{4,4,2,2\} \text{ in some order} \,\middle|\, \sum X_i = 12\right).$$

The major task is to list the restricted possibility space in a systematic way. Classify by the largest number, starting a table as follows:

Largest	Others	Orders
3	$\{3,3,3\}$	1
4	$\{4,3,1\}$	12
4	$\{4,2,2\}$	?
4	?	?
...	...	...

4834. By symmetry, the area enclosed by the curves is twice the area enclosed by $y=x^2$ and $y=\sqrt{3}x$.

4835. When writing an expression in harmonic form, the usual method is to take the primary solutions of both R and α. With equations $R\sin\alpha = a$ and $R\cos\theta = b$, these are

- $R = +\sqrt{a^2+b^2}$,
- $\alpha = \arctan\frac{a}{b}$.

But the fact that such values are *individually* valid doesn't necessarily mean that they *combine* to make a valid solution.

4836. Find the coordinates of the stationary points, and show that they are symmetrical. Since there are three of them, the rest of the curve must follow. For further proof, use the substitution $z = x - a$, where $x = a$ is the line of symmetry.

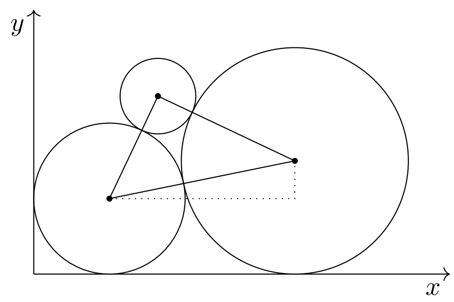
4837. Assume, for a contradiction, that there exists an irregular pentagon, with perimeter P, which has maximal area. Let a and b be the lengths of two adjacent sides, where $a \neq b$. Show that, by moving the vertex in between these two sides, it is possible to increase the area of the pentagon.

4838. (a) Use the quotient rule.

 (b) The higher derivatives describe how quickly the first derivative changes. So, if the higher derivatives are all approximately zero, then the first derivative changes very slowly.

 (c) Use parts (a) and (b), together with the fact that the graph has period 2π. The curve has the shape of a sawtooth wave.

4839. Rotate the picture so that the two larger circles are tangent to the x axis:



Consider a rectangle with sides parallel to the axes in this picture.

4840. The string is smooth, so, by symmetry, the lines of action of the three forces lie on the angle bisectors of the triangle of string, meeting at the incentre. Work in a diagram of the triangle of string: you can locate the angles in the triangle of forces on that diagram.

4841. Let Y be the integral of y. You can then rewrite the integral equation as a differential equation in Y and x. Solve this by separation of variables, using a trig substitution. Then differentiate to find y. Take care to introduce any constant of integration at the correct moment.

4842. Edge V_1V_2 has two endpoints, from each of which there are two edges besides V_1V_2. Having chosen V_1V_2 wlog, there are five edges which cannot be chosen as V_3V_4. There are seven edges remaining in the possibility space. Only one of these has the potential to be successful. Work out which one, and then consider the ${}^4\mathrm{C}_2 = 6$ ways of selecting the remaining two vertices.

4843. The curves are a circle, centre $(0, 6)$, radius 4, and a parabola. The shortest distance between them lies along the normal to both. This is an extended radius of the circle, and must pass through $(0, 6)$.

4844. The form for the partial fractions is

$$\frac{1}{x^2(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2}.$$

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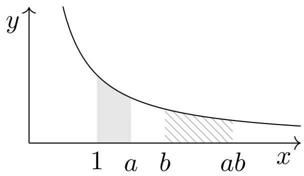
4845. You could do this by explicit chain rule, but it's tricky, as the SPs aren't at nice values. You're better off taking a graphical approach. Show that $y = \text{gh}(x)$ has three distinct roots, by looking at its SPs. Then consider the effect of applying f.

4846. The pegs are smooth, so the tension is the same throughout. Hence, the resultant force applied by the string at A acts along the angle bisector at A. Let θ be half the angle at A.

Use the cosine rule to write $\cos 2\theta$ in terms of the side lengths. Then use a double-angle formula to find $\cos\theta$. Resolving along the angle bisector.

4847. (a) Draw the standard reciprocal graph.

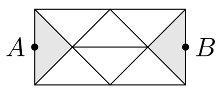
(b) Consider two transformations which transform the solid region into the hatched region:



4848. Differentiate $y = ux$ with respect to x. Sub in and simplify to a separable DE in variables u and x. Solve this, then reverse the substitution.

4849. The index 5.5 is $11/2$. The numerator is odd, which maintains signs. The denominator is even, so the curve has no points where $\cos x < 0$. Consider the effect of raising to the power 5.5 the points on $y = \cos x$ where $y = 0$ and where $y = 1$.

4850. The regions containing A and B must be shaded. So, with probability $\frac{1}{4}$, start with



There are now $2^6 = 64$ outcomes. Classify these according to n, the number of shaded regions.

4851. The mass is 1, so the force gives the acceleration directly. The acceleration is 1 ms^{-2} for 1 second, then 2 ms^{-2} for 1 second, and so on, until it is n ms^{-2} for 1 second. Thereafter, the object moves at constant velocity.

(a) $\Delta v = 1 + 2 + 3 + ... + n$.

(b) The displacement is given by the area under a velocity-time graph. So, for $t \in [0, n)$, you need to calculate

$$\sum_{r=1}^{n} \tfrac{1}{2} r(r+1).$$

Do this by multiplying out and using the given result. Then consider $t \in [n, 2n]$.

4852. Take out a factor of $(x - 1)$ from top and bottom. Cancel this, then take the limit.

4853. This can be proved either graphically, using area of a rectangle, or by parts. These are, in fact, two ways of looking at the same method.

4854. Consider the divisibility of the number between the primes.

4855. Classified by the number n of sides of the square to which both circles are tangent, the cases are

$$r = \begin{cases} \left(3 - 2\sqrt{2}\right)R, & n = 2 \\ \dfrac{2 - 2R}{1 + \sqrt{2}}, & n = 0 \\ \left(1 - \sqrt{R}\right)^2, & n = 1. \end{cases}$$

4856. Differentiate to show that $\text{E}'(x) = \text{E}(x)$. Then let $y = \text{E}(x)$, and solve as a separable DE in x and y. Find the particular solution by substituting $x = 0$.

4857. Substitute the parametric equations into the LHS of the equation of the ellipse. You need to find a value of the parameter t for which this expression exceeds 9. Optimise using calculus.

4858. Let the sequences be

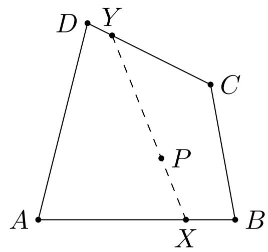
$$\{u_n\} = \{a, a + d, a + 2d, ...\}$$
$$\{v_n\} = \{a, ar, ar^2, ...\}$$

Set up simultaneous equations, and substitute for d, solving to find a simple relationship between a and r. Use the fact that a and r are integers to find the only possible value of a.

4859. Let points X and Y have position vectors

$$\overrightarrow{OX} = \frac{\lambda_1\mathbf{a} + \lambda_2\mathbf{b}}{\lambda_1 + \lambda_2},$$
$$\overrightarrow{OY} = \frac{\lambda_3\mathbf{c} + \lambda_4\mathbf{d}}{\lambda_3 + \lambda_4}.$$

According to the assumed result, you know that X lies on AB and Y lies on CD.



Write $\overrightarrow{OP}$ in terms of $\overrightarrow{OX}$ and $\overrightarrow{OP}$, and thereby show that P lies on the chord XY.

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4860. Rearrange to $\tan y = \cos x$ and then differentiate implicitly. Use the second Pythagorean identity and then sub $\tan y = \cos x$ back in.

4861. (a) The greatest triangle is equilateral.
(b) The greatest rectangle is not a square.
(c) The greatest hexagon is regular.

4862. (a) The relevant fact is the parity of the degree of the polynomials f and f$''$.
(b) Consider a linear function f.

4863. Find the equation of the horizontal asymptote by dividing top and bottom by x^2. Then find the equation of a generic normal to the curve at $x = a$. Substitute the centre of the circle and solve.

4864. Rewrite the integrand as

$$\frac{a^2 + x^2}{b^2 + x^2} \equiv 1 + \frac{a^2 - b^2}{b^2 + x^2}.$$

Then use the substitution $x = b \tan \theta$.

4865. Use the generalised binomial expansion on the equation of a semicircle.

— Alternative Method —

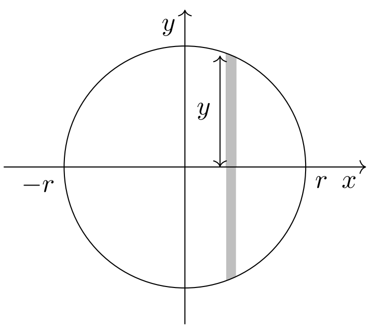
Compare zeroth, first and second derivatives.

4866. The central triangle is equilateral, as are the three triangles formed outside it. Find the length of the chord by considering the angle subtended at the centre. Subtract $2l$, where l is the side length of the regular nonagon. Hence, show that the side length is

$$2 \sin 80° - 4 \sin 20°.$$

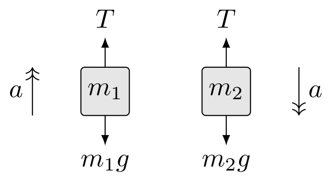
Use identities to manipulate this.

4867. Consider the sphere in cross-section:

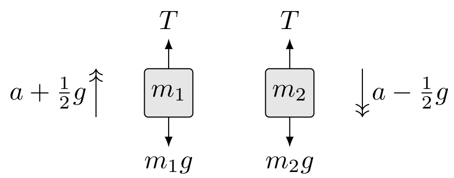


The circle has equation $y^2 = r^2 - x^2$. So, the shaded strip shown can be seen as the cross-section of (approximately) a disc of area $y^2 = \pi(r^2 - x^2)$. Use a definite integral to sum this quantity over the domain $x \in [-r, r]$.

4868. (a) If the pulley does not accelerate, then it acts like a fixed pulley. The force diagrams are



(b) If the pulley accelerates upwards at $\frac{1}{2}g$, then the forces are as before (although the value of T has changed). The accelerations of the bobs now depend on both a (acceleration of the string around the pulley), and also the $\frac{1}{2}g$ acceleration of the pulley:



4869. (a) Use Pythagoras. The algebra is easier if you work with the *squared* distance.
(b) Follow the same argument as in (a).
(c) Add your results.

4870. Consider a network with n users. Assume, for a contradiction, that no two users have the same number of connections. The smallest possible number of connections is 0 and the greatest is $n-1$. So, there are n possible numbers of connections:

$$\{\underbrace{0, 1, 2, ..., n-1}_{n \text{ numbers}}\}.$$

Consider the allocation of these numbers among the users, and seek out a contradiction.

4871. Since $|\sin x|$ and $|\sin y|$ are periodic, period π, you need only consider the square domain $[0, \pi] \times [0, \pi]$.

4872. (a) Set the derivative to zero and solve.
(b) A cubic must have at least one root. Consider an approximation to the equation when x is large. Then use the SPs from part (a) to show that there are no other real roots.

4873. To find the rate of change of θ, you need to look at two elements:

- velocity perpendicular to OF,
- the distance $|OF|$.

Find the ratio in each case, then work out how each scales ω.

4874. Use integration by parts, but not with either of the obvious choices for u and v'. You're heading for integration by inspection.

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4875. Firstly, sketch the parametric curve

$$x = 2\sin 2t,$$
$$y = 2\sin t.$$

This is the path of the centre of the circle.

4876. Use a prime factor argument. Starting with 2 and working upwards, consider which and how many prime factors can be *guaranteed* to be present.

4877. Let $y = \text{g}_a(x)$ be the equation of the tangent at $x = a$. The equation for intersections is

$$x^3 + 3x^2 - 5 - \text{g}_a(x) = 0.$$

Consider the multiplicity of the roots.

4878. Find the equation of the tangent at O. Then solve for intersections of this tangent with the original curve, and show that the resulting equation has a triple root, i.e. that the tangent crosses the curve.

4879. For fixed points, $\displaystyle\sum_{r=1}^{2k+1} x^r = x.$

Write this equation longhand, and take out, one by one, the factors corresponding to two roots of the equation/fixed points of I. Then show that the remaining factor has no real roots.

4880. (a) Find the gradient of the common tangent.

(b) Set up a vertical equation of motion for the cylinder and a horizontal equation of motion for the half-cylinder.

4881. (a) Find $\frac{dy}{dx}$ and substitute in.

(b) Use the substitution $x = r\sin\theta$.

4882. Use a combinatorial method. The possibility space consists of the orders of

$$\{[\text{RRR}], \text{B}, \text{B}, \text{B}, \text{G}, \text{G}, \text{G}\}.$$

In successful outcomes, the blue counters also form a group:

$$\{[\text{RRR}], [\text{BBB}], \text{G}, \text{G}, \text{G}\}.$$

Work out the numbers of orders of each of these.

4883. (a) These results occur because both terms are squares, which are non-negative.

(b) Differentiate implicitly and set $\frac{dy}{dx} = 0$. Sub the resulting equations back into the curve, giving four SPs.

(c) The curve is a closed loop around the origin, as seen from (a). Find the x and y intercepts. Together with the stationary points from (b), this should allow you to join the dots.

4884. (a) Let the speed of the wheel be 1 unit/s. The coordinates of the centre C are then $(t, 1)$. The radial vector rotates clockwise at 1 radian per second, starting at $-\mathbf{j}$. Express this in terms of t, then combine as

$$\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP}.$$

(b) Set things up with the parametric integration formula, and then use a double-angle formula to carry out the integral.

4885. Find all SPs and sketch $y = \text{f}(x)$ carefully, noting the proximity of some roots and SPs. Calculate the equation of the tangent at $x = 3$, and add this to your sketch. Consider graphically the running of the N-R iteration. You should find three possible behaviours for $x_0 \geq 3$.

4886. Simplify the first bracket on its own. Multiply by the second bracket and split up the fraction.

4887. The sum S is given by $S = S_1 - S_p - S_q + S_{pq}$:

- S_1 is the sum of the first npq integers,
- S_p is the sum of those divisible by p,
- S_q is the sum of those divisible by q,
- S_{pq} is the sum of those divisible by p and q.

4888. (a) Draw a force diagram for one of the cards, and take moments around the base.

(b) Draw separate force diagrams for each card. Calling the length 2, take moments around the base of each, using the ratios $\sin\theta = \frac{12}{13}$ and $\cos\theta = \frac{5}{13}$ to simplify as you go.
Assume limiting friction at the top, giving $F_{\text{max}} = \mu R_{\text{top}}$. You should end up with two equations in μ, R_{top} and W. Eliminate R_{top} and W will go too. Solve for μ.

4889. Let $a_n = ar^{n-1}$ and $b_n = bs^{n-1}$. Set up

$$\frac{a_3 + b_3}{a_2 + b_2} = \frac{a_2 + b_2}{a_1 + b_1}.$$

Solve this to show that $r = s$.

4890. Let A start. Restrict the possibility space to the round in which the competition is decided; this will save you finding infinite sums. Set up the following equation:

$$\mathbb{P}(A \text{ wins} \mid \text{a result}) = \mathbb{P}(B \text{ wins} \mid \text{a result}).$$

Solve by factorising.

4891. Set up the equation of a generic normal at (a, a^2). Find the y intercept of this normal. Use it to find the area of the shaded region: a trapezium minus the region below the curve. Set this to 1.452.

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4892. Rename x as m, and then consider $\arctan m$ as the angle of inclination of the line $y = mx$.

——— ALTERNATIVE METHOD ———

Let $a = \arctan x$ and $b = \arctan {}^{1}/_{x}$. Consider $\tan(a + b)$. Expand this with a compound-angle formula, and consider the fact that the resulting expression is undefined.

4893. Factorise to find the x intercepts in each case, and consider the multiplicity of the roots. Then set up the equation $x^4 - x^2 = x^6 - x^2$ for intersections of the curves: apply the same process to this.

4894. Find expressions for $\sin 27° \pm \cos 27°$ by squaring each and simplifying. Once you have the pair of expressions, add them to eliminate $\cos 27°$.

4895. Derive the relevant product-to-sum formula:

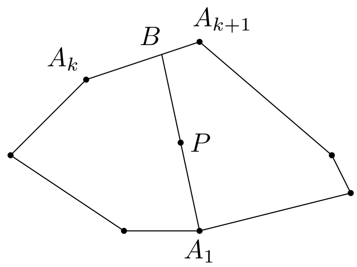
$$\begin{aligned} &\cos(p+q) - \cos(p-q) = 2\sin p \sin q \\ \Longrightarrow\ &\sin p \sin q = \tfrac{1}{2}\big(\cos(p+q) + \cos(p-q)\big). \end{aligned}$$

Then find the integral directly. To evaluate the trig functions, you'll need to use the fact that a and b are natural numbers.

4896. Prove this by construction. Draw a line through P and vertex A_1. If this line passes through another vertex A_k, then you have the result immediately: a point on A_1A_k can be written as

$$\mathbf{p} = \lambda \mathbf{a}_1 + (1 - \lambda)\mathbf{a}_k.$$

Otherwise, the line passes through an edge. Call this A_kA_{k+1} and the point of intersection B.



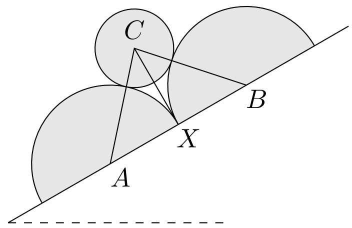
Express $\mathbf{b}$ in terms of $\mathbf{a}_k$ and $\mathbf{a}_{k+1}$. Also, express $\mathbf{p}$ in terms of $\mathbf{a}_1$ and $\mathbf{b}$. You'll need two parameters. Combine the two expressions.

4897. Set up the LHS of the proposed equation on its own. Take out a common factor of $\frac{1}{4}(n+1)^2$. Simplify the remaining quartic factor and factorise it fully. Start again with the RHS, taking out a factor of $\frac{1}{4}(n+1)^2(n+2)^2$ to reach the same expression.

4898. The possibility space consists of all outcomes with rotational symmetry. Count these, classifying by order 2 or order 4. Be careful not to double count the order 4 outcomes as order 2.

4899. Use $u = \cos x$, then partial fractions.

4900. (a) The scenario is



Triangle ABC has side lengths $(3r, 3r, 4r)$, so

$$\angle ACX = \angle BCX = \arcsin {}^{2}/_{3}.$$

Work from there.

(b) Set up the sine rule:

$$R_1 = \frac{\sin(\arcsin {}^{2}/_{3} + 30°)}{\sin(180° - 2\arcsin {}^{2}/_{3})} W.$$

To keep the algebra manageable, look at top and bottom separately. Use double-angle and then compound-angle formulae to simplify.

——— END OF 49TH HUNDRED ———

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